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NATURAL VIBRATIONS OF LAMINATED ORTHOTROPIC SPHERES

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Abstract—A method based on the Rayleigh–Ritz technique is used to study the natural vibrations of elastic spheres composed of an arbitrary number of spherical layers, each with distinct density and spherically orthotropic (transversely isotropic) properties. The basic form of the displacement field known from studies of isotropic spheres is employed, except for the radial dependence which is taken in an approximate manner by considering the sphere to be an assemblage of a large number of spherical laminas; for each lamina, the radial dependence of the displacement field is characterized by a discrete number of generalized coordinates. The approximation results in two algebraic eigenvalue problems which are solved for examples of isotropic and laminated orthotropic spheres.

INTRODUCTION

THE natural vibrations of elastic isotropic spheres have been studied for nearly a century, from the early works of Jaerish [1] and Lamb [2], to the recent analyses given by Sato and Usami [3] and Shah *et al.* [4]. In the latter paper the natural vibrations of spheres were used to examine the accuracy of several shell theories in a manner similar to Mindlin's [5] comparison of exact and approximate plate theories and to Mirsky's [6] study of circular cylinders. The behavior of laminated media in general and particularly laminated spheres has received little attention, except for approximate analyses [7–9], or for cases of homogeneous orthotropic or simple two layer systems with isotropic laminates [6, 10–14]. Recently, analyses were presented of the natural vibrations and waves in arbitrarily laminated orthotropic infinite plates [15] and circular cylinders [16].

In this paper a Rayleigh–Ritz procedure analogous to that given in [15, 16] is used to investigate the natural vibrations of elastic spheres composed of an arbitrary number of laminates, each with specified density and spherically orthotropic, i.e. transversely isotropic material properties. In this procedure the dependence of the displacement field in the spherical angles θ and ϕ is specified at the outset as in the classical approach, but the radial dependence is taken in an approximate manner using a discrete set of generalized coordinates. This approach results in an algebraic eigenvalue problem which is solved by use of an efficient eigensolution technique [17]. The accuracy and efficiency of this formulation are demonstrated by reproducing the results given by Sato and Usami [3] for a solid isotropic sphere and an additional example is presented to indicate its versatility and range of applicability.

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FORMULATION OF THE FREQUENCY EQUATION

In order to study the vibrations of laminated orthotropic spheres the displacement forms which describe the behavior of isotropic spheres are required. As given in [4], the displacement forms u_i, $i = r, \theta, \phi$ are of two types,

(a) *First class vibrations* (*equivoluminal*)

$$
\begin{aligned}
u_r(r, \theta, \phi, t) &= 0 \\
u_\theta(r, \theta, \phi, t) &= mU_1(r, t)P_n^m(\cos\theta)\sin m\phi \\
u_\phi(r, \theta, \phi, t) &= U_1(r, t)\frac{\mathrm{d}P_n^m}{\mathrm{d}\theta}(\cos\theta)\cos m\phi.
\end{aligned}
\tag{1a}
$$

(b) *Second class vibrations*

$$
\begin{aligned}
u_r(r, \theta, \phi, t) &= U_2(r, t)P_n^m(\cos\theta)\cos m\phi \\
u_\theta(r, \theta, \phi, t) &= U_3(r, t)\frac{\mathrm{d}P_n^m}{\mathrm{d}\theta}(\cos\theta)\cos m\phi \\
u_\phi(r, \theta, \phi, t) &= -mU_3(r, t)\frac{P_n^m(\cos\theta)}{\sin\theta}\sin m\phi
\end{aligned}
\tag{1b}
$$

where t denotes time, r the spherical radius, θ and ϕ spherical angles, see Fig. 1 and P_n^m denotes the associated Legendre function of degree n and order m with m, n positive integers. Although the analytical development in [4] is presented for isotropic spheres, the forms given in equations (1) also apply for the case of transverse isotropy, as may be verified by direct substitution into the displacement equations of motion. Further, substitution also shows the functions $U_i(r, t)$, $i = 1, 2, 3$ to be independent of the value m. This occurrence, together with the fact that the value m does not influence the interface or free boundary conditions, leads to a frequency equation and thus frequencies and radial distributions of the associated eigenvectors which are independent of m. Consequently only the case $m = 0$ need be considered where the solution forms are much simpler.

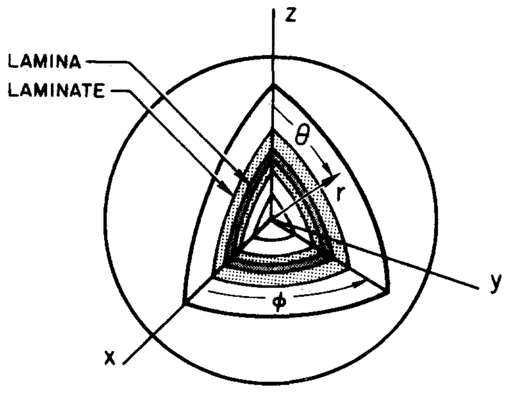

FIG. 1. Laminated orthotropic sphere.

(a) *First class vibrations* (*torsional*)

$$\begin{aligned} u_r &= u_\theta = 0 \\ u_\phi &= U_1(r,t)\frac{\mathrm{d}P_n}{\mathrm{d}\theta}(\cos\theta). \end{aligned} \tag{2a}$$

(b) *Second class vibrations*

$$\begin{aligned} u_r &= U_2(r,t)P_n(\cos\theta) \\ u_\theta &= U_3(r,t)\frac{\mathrm{d}P_n}{\mathrm{d}\theta}(\cos\theta) \\ u_\phi &= 0 \end{aligned} \tag{2b}$$

with $P_n \equiv P_n^0$ a Legendre polynomial of order n. Rather than directly solve the field equations for the radial dependence in the functions $U_i(r, t)$ $i = 1, 2, 3$ and formulate the frequency equation in the usual way, the sphere is represented by a number of concentric spherical subregions called laminas, several of which may represent a single laminate. For each lamina the quantities $U_i(r, t)$ in equation (2) are approximated for convenience by the quadratic form

$$U_i(r,t) = U_{Ii}(t)\,.\,(1-3\hat{r}+2\hat{r}^2)+U_{Mi}(t)\,.\,(4\hat{r}-4\hat{r}^2)+U_{Oi}(t)\,.\,(-\hat{r}+2\hat{r}^2) \quad i = 1,2,3 \tag{3}$$

where

$$\hat{r} = (r-r_I)/h \tag{4}$$

is a local radial variable, r_I is the inner radius of the lamina, h its thickness, and U_{Ii}, U_{Mi} and U_{Oi} are the generalized coordinates for inner, middle and outer lamina surfaces, respectively.

The strain and kinetic energies for the lamina are obtained by integrating the strain and kinetic energy densities over the lamina volume

$$\begin{aligned} V = \frac{1}{2}\int_0^{2\pi}\int_0^{\pi}\int_{r_I}^{r_I+h} & [C_{11}\varepsilon_{rr}^2 + 2C_{12}\varepsilon_{rr}(\varepsilon_{\theta\theta}+\varepsilon_{\phi\phi}) + C_{22}(\varepsilon_{\theta\theta}^2+\varepsilon_{\phi\phi}^2) \\ & + 2C_{23}\varepsilon_{\theta\theta}\varepsilon_{\phi\phi} + 4C_{44}(\varepsilon_{r\theta}^2+\varepsilon_{r\phi}^2) + 2(C_{22}-C_{23})\varepsilon_{\theta\phi}^2]r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \end{aligned} \tag{5a}$$

$$T = \frac{1}{2}\int_0^{2\pi}\int_0^{\pi}\int_{r_I}^{r_I+h} \rho(\dot{u}_r^2+\dot{u}_\theta^2+\dot{u}_\phi^2)r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \tag{5b}$$

where ε_{ij} and C_{ij} are, respectively, the small strain tensor components and the transversely isotropic elastic moduli, ρ is the mass density and the dot denotes differentiation in time. After substitution of the displacement forms in equations (5) and integration, the energies take the form

$$V^i = \tfrac{1}{2}\{r^i\}^T[k^i]\{r^i\} \qquad i = 1,2 \tag{6a}$$

$$T^i = \tfrac{1}{2}\{\dot{r}^i\}^T[m^i]\{\dot{r}^i\} \tag{6b}$$

where the superscript $i = 1, 2$ denotes the ith vibration class and the symbol $\{r^i\}$ is one of the ordered sets of coordinates

$$\{r^1\}^T = \{U_{I1}, U_{M1}, U_{O1}\} \tag{7a}$$

$$\{r^2\}^T = \{U_{I2}, U_{I3}, U_{M2}, U_{M3}, U_{O2}, U_{O3}\} \tag{7b}$$

and $[k^i]$ and $[m^i]$ are the lamina stiffness and mass matrices, respectively, for the ith vibration class. The Lagrangian L for the complete system is obtained by summation over all the laminas

$$L = \sum (T^i - V^i) = \tfrac{1}{2}\{\dot{U}^i\}^T[M^i]\{\dot{U}^i\} - \tfrac{1}{2}\{U^i\}^T[K^i]\{U^i\} \tag{8}$$

where $\{U^i\}$ is the set of generalized coordinates and $[K^i]$ and $[M^i]$ are the stiffness and mass matrices for the ith class vibrations of the entire sphere. Use of Lagrange's equations gives

$$[K^i]\{U^i\} + [M^i]\{\ddot{U}^i\} = 0 \tag{9}$$

which, for simple harmonic motion

$$\{U^i\} = \{U_0^i\}\, e^{i\omega t} \tag{10}$$

reduces to the eigenvalue problem

$$([K^i] - \omega^2[M^i])\{U_0^i\} = 0. \tag{11}$$

Although this equation can be analyzed by a variety of numerical techniques for this study a direct iterative eigensolution technique [17] is used, since a number of eigenvalues and associated eigenvectors are simultaneously generated. An IBM 360/91 computer was programmed to generate for each value n sets of eight and sixteen eigensolutions, respectively, for first and second class vibrations. The algorithm proved quite efficient, generating 24 eigenvalues and eigenvectors for 15 values n in less than two minutes.

Since the eigenvectors (the U_0^i distributions) are automatically generated by the solution procedure, the stresses in each lamina can easily be obtained by use of strain–displacement and constitutive relations.

EXAMPLES

In order to briefly indicate the accuracy and versatility of this method two examples are presented. In both examples the radius R (the average of the inner and outer surface radii) and the maximum material density in the laminates are taken as unity, and 50 equal thickness laminas are chosen to model the sphere, resulting in 101 and 202 components for the vectors $\{U_0^1\}$ and $\{U_0^2\}$, respectively.

Solid isotropic sphere

Results for the lowest natural frequencies of a solid isotropic sphere (Poisson's ratio $\nu = 0{\cdot}25$) obtained by the present method and those given by Sato and Usami [3] are listed in Table 1. The reference frequency ω_{Ref} is taken as $a^{-1}(\mu/\rho)^{\frac{1}{2}}$ with a the sphere radius and μ the shear modulus. As is evident, the results of the present method are in excellent agreement with those in [3] for all values n considered. These results could have been improved even further by using a more elaborate model.†

† For example the fifth lowest frequency at $n = 50$ for second class vibration is 74·833 for the 50 lamina model and 74·818 for a 75 equal thickness lamina model as compared to Sato and Usami's value 74·814.

TABLE 1. COMPARISON OF FREQUENCIES $\Omega = \omega a/(\mu/\rho)^{\frac{1}{2}}$ FOR SOLID ISOTROPIC SPHERE; $\nu = 0{\cdot}25$

	Mode No. 1		Mode No. 2		Mode No. 3		Mode No. 4		Mode No. 5	
Values n	Present results	Ref. [3]	Present results	Ref. [3]	Present results	Ref. [3]	Present results	Ref. [3]	Present results	Ref. [3]
First class vibrations†										
1	‡	‡	5·763	5·763	9·095	9·095	12·323	12·322	15·515	15·514
2	2·501	2·501	7·136	7·136	10·515	10·514	13·772	13·771	16·983	16·983
5	6·266	6·266	10·951	10·950	14·511	14·510	17·886	17·885	21·181	21·180
10	11·792	11·792	16·882	16·882	20·732	20·731	24·311	24·310	27·760	27·760
25	27·554	27·554	33·644	33·643	38·159	38·158	42·245	42·244	46·106	46·104
50	53·240	53·240	60·475	60·474	65·762	65·760	70·466	70·461	74·852	74·842
100	104·067	104·066	112·845	112·841	119·191	119·179	124·769	124·743	129·919	129·871
Second class vibrations										
0	4·440	4·440	10·494	10·494	16·073	16·073	21·579	21·579	27·059	—
1	‡	‡	3·425	3·424	6·771	6·771	7·745	7·744	10·695	10·695
2	2·640	2·640	4·865	4·865	8·329	8·329	9·780	9·780	12·157	12·157
5	6·033	6·033	9·636	9·636	12·368	12·368	15·179	15·179	16·818	16·818
10	10·855	10·855	16·619	16·619	19·282	19·282	22·080	22·080	24·886	24·886
25	24·834	24·833	33·615	33·615	38·112	38·111	41·883	41·882	45·049	45·046
50	47·926	47·912	60·184	60·183	65·708	65·704	70·471	70·461	74·833	74·814
100	94·268	—	112·098	—	118·868	—	124·638	—	129·904	—

† No nontrivial results exist for $n = 0$.
‡ Rigid body motion.

Laminated orthotropic sphere

In order to give an indication of the versatility of the method a three layer hollow sphere with $H/R = 1{\cdot}0$ is considered. The inner and outer layers of thickness $0{\cdot}3H$ and $0{\cdot}2H$, respectively, are composed of a material with properties

$$C^1_{11} = 0{\cdot}70 \quad C^1_{12} = 0{\cdot}30 \quad C^1_{22} = 7{\cdot}00 \quad C^1_{23} = 3{\cdot}00 \quad C^1_{44} = 2{\cdot}00 \quad \rho^1 = 1{\cdot}0$$

and the middle layer of thickness $0{\cdot}5H$ is composed of a material with properties

$$C^2_{11} = 0{\cdot}70 \quad C^2_{12} = 0{\cdot}30 \quad C^2_{22} = 0{\cdot}70 \quad C^2_{23} = 0{\cdot}30 \quad C^2_{44} = 0{\cdot}02 \quad \rho^2 = 1{\cdot}0$$

where the superscript denotes material number. Letting $\omega_{\mathrm{Ref}} = \pi/H(C^1_{44}/\rho^1)^{\frac{1}{2}}$, a list of frequencies is given in Table 2 and several displacement distributions are shown in Fig. 2,

TABLE 2. LOWEST FIVE FREQUENCIES $\Omega = \omega H/\pi(C^1_{44}/\rho^1)^{\frac{1}{2}})$ FOR FIRST AND SECOND CLASS VIBRATIONS OF LAMINATED ORTHOTROPIC SPHERE

	Mode No. 1		Mode No. 2		Mode No. 3		Mode No. 4		Mode No. 5	
Values n	First class	Second class	First class	Second class	First class	Second class	First class	Second class	First class	Second class
0	–	0·5888	–	1·1316	–	1·6609	–	2·0603	–	2·5662
1	†	†	0·1393	0·3036	0·2769	0·3420	0·4479	0·4925	0·6332	0·6351
2	0·2697	0·2571	0·3971	0·4931	0·4913	0·5281	0·6362	0·7038	0·7997	0·7296
5	0·5368	0·4778	0·6557	0·8299	0·7911	0·9659	0·9442	1·0958	1·0978	1·1214
10	0·9580	0·8358	1·0831	1·3149	1·1942	1·5546	1·3083	1·7061	1·4363	1·8172
25	2·1726	1·2211	2·3314	2·0041	2·4663	2·7550	2·5892	3·2891	2·7045	3·5310

† Rigid body motion.

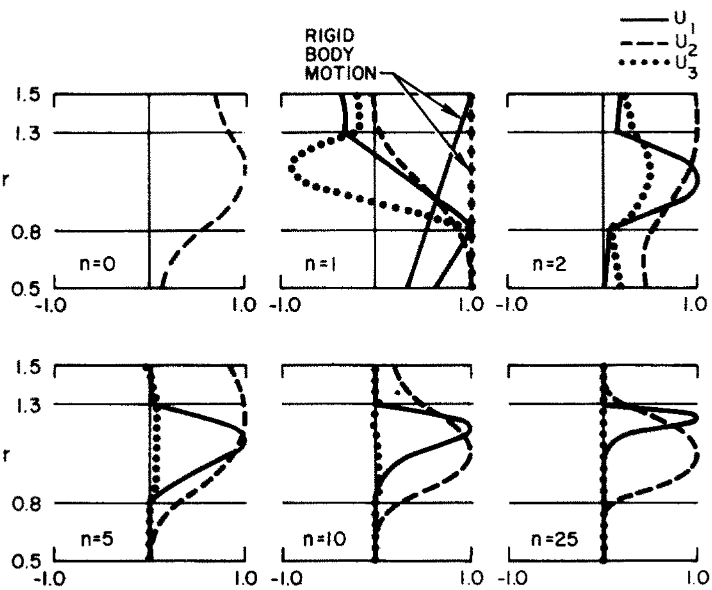

FIG. 2. Displacement distributions in r of lowest vibratory modes for laminated sphere.

clearly indicating the significant changes in the physical behavior of the laminated sphere. Of special interest are the cases $n > 25$, where nearly all motions occur in the weaker interior layer.

CONCLUSIONS

An efficient method has been presented for studying the natural vibrations of arbitrarily laminated orthotropic spheres. The method is shown to give accurate results over a large range of frequencies and values n and to automatically furnish information on the radial dependence of the displacements. The results of this analysis, along with those for laminated plates [15] and cylinders [16] will be useful for determining modeling capabilities of various structural models and will provide a reference for developing refined plate and shell theories. The results obtained may also be used to study the response of spheres to transient loads, a problem of interest in biomechanical and geophysical applications.

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Абстракт—Применяется метод, основанный на способе Рейлея—Ритца, с целью исследования свободных колебаний упругих сфер изтовленных из произвольного числа сферических слоев, каждый из которых обладает разной плотностью и разными свойствами сферической ортотропии/поперечная изотропия/. Применяется основная форма для поля перемещений, известная из изучения изотропной сферы, но за исключением радиальной зависимости, которую обсуждается приближенным способом, рассматривая сферу как совокупность большого числа сферических слоев. Для каждого слоя, радиальная зависимость поля перемещений характеризуется конечным числом обобщенных координат. Даются приближенные решения для двух алгебраических задач на собственные значения, расчитанных для изотропной и слоистой ортотропной сферы.